Test 1 - Math Thought

Dr. Graham-Squire, Spring 2016

Name: ________________________________

I pledge that I have neither given nor received any unauthorized assistance on this exam.

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(*signature*)

DIRECTIONS

(1) Don't panic.

(2) Show all of your work and use correct notation. A correct answer with insufficient work or incorrect notation will lose points.

(3) You are required to do the first 5 questions on the test. For questions 6 through 10, you only need to do three of the questions. It is fine if you do all five of the questions 6-10, though–I will grade them all and just give you the points for the top 3 scores.

(4) Cell phones and computers are not allowed on this test. Calculators are allowed, though it is unlikely that they will be helfpul.

(5) Make sure you sign the pledge above.

(6) Number of questions = 8 to 10. Total Points = 40.

(1) (4 points) Find truth values for P, Q, R and S to show that $[(P \to Q) \to R] \to S$ and $P \to [Q \to (R \to S)]$ are not logically equivalent.

(2) (4 points) A sequence of real number $\{x_1, x_2, x_3, \dots\}$ is called a BooYaa sequence provided that for each positive real number a, there exists a positive integer N such that for all $z \in \mathbb{Z}^+$, if $z > N$, then $|x_z| < a$.

(a) Use mathematical notation to express what it means to be a BooYaa sequence.

(b) Use words to carefully explain what it means for a sequence to NOT be a BooYaa sequence.

(3) (4 points) Let the domain be $\mathbb{R}$ =real numbers. Are the following statements true or false? Are they logically equivalent? Explain why or why not.

(i) $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(\forall z \in \mathbb{R})(x + y - z = 0)$

(ii) $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})(\exists z \in \mathbb{R})(x + y - z = 0)$

(4) (8 points) Let n be an integer. Use the *definition* of even and/or odd to prove the following:

$$n + 3 \text{ is even} \leftrightarrow n^2 + 3 \text{ is even.}$$

(5) (5 points) For the following statement/proof, you must choose one of two options:

(a) The statement is false, and thus the proof is false. In this case, you must say how/where the proof is false and then give a counterexample or explanation to show how the statement is false.

(b) The statement is true, but the proof is false and/or poorly written. In this case, you must say how/where the proof is false or what about it is poorly written, and then write a correct proof.

The statement: "If $n \not\equiv 2 \pmod{10}$, then $n^2 \not\equiv 4 \pmod{10}$."

Proof. Proof by contrapositive. Suppose that $n \equiv 2 \pmod{10}$. Then by the definition and properties of equivalence modulo 10, we have

$$\begin{array}{rrcll} & n & = & 2+10k & \text{for some } k \in \mathbb{Z} \\ \rightarrow & n^2 & = & (2+10k)^2 & \text{(squaring both sides)} \\ \rightarrow & n^2 & = & 4+40k+100k^2 & \text{(math)} \\ \rightarrow & n^2 & = & 4+10(4k+10k^2) & \text{(factoring)} \\ \rightarrow & n^2 & \equiv & 4 \pmod{10} & \text{(definition of congruence mod 10)} \end{array}$$

Thus we have shown that if $n \equiv 2 \pmod{10}$, then $n^2 \equiv 4 \pmod{10}$, so by the contrapositive we have "If $n \not\equiv 2 \pmod{10}$, then $n^2 \not\equiv 4 \pmod{10}$," as desired. $\square$

For questions 6 to 10, you only need to answer 3 out of the five questions. If you answer all of them, that is fine–I will grade all of them and give you the highest 3 scores from the 5 questions.

(6) (5 points) Is the following statement true or false? If true, prove it. If false, find a counterexample.

$$\text{For all integers } x \text{ and } y,\ (x+y)^2 \equiv (x^2+y^2) \pmod 2.$$

(7) (5 points) Suppose that $a, b, c \in \mathbb{Z}$ and $c \neq 0$. Use the definition of "divides" to prove that if ac divides bc, then a divides b.

(8) (5 points) Prove that if m is an integer, then 3 divides $m^2 - m$ or 3 divides $m^2 - m - 2$.

(9) (5 points) Use the *definitions* of rational and/or irrational to prove the following:

For all nonzero real numbers x and y, if x is rational and y is irrational, then $\frac{x}{y}$ is irrational.

(10) (5 points) Prove the following statement. You will need to use the Pythagorean theorem for right triangles.

Suppose that j and m are positive integers, j and m are the lengths of the legs of a right triangle, and $m+1$ is the length of the hypotenuse of the triangle. Prove that j must be an odd integer.

Extra Credit(2 points) Suppose $P(x)$ and $Q(x)$ are open sentences. Is

$$\Big((\exists x \in \mathbb{R})\big(P(x) \leftrightarrow Q(x)\big)\Big) \equiv \Big((\exists x \in \mathbb{R})\big(P(x)\big) \leftrightarrow (\exists x \in \mathbb{R})\big(Q(x)\big)\Big)?$$

Explain your answer.